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Analytic properties of the partition function for statistical mechanical models

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Abstract. We explain, with examples, the relationship between zeros of the partition function, analyticity and degeneracy of absolute magnitude of eigenvalues of the transfer matrix for statistical mechanical models. We show how to write down a polynomial representation of the partition function for any model with a two-sheet largest eigenvalue. We solve the staggered ice model representation of the q-state Potts model on a sequence of semi-infinite strips and show that for $0<q<4$ the Potts model result is only projected out by special boundary conditions. The non-Potts part of the result is obtained for a $4\times\infty$ strip and gives Baxter's antiferromagnetic critical curve.

1. Introduction

There have been many papers discussing the distribution of zeros of the partition function in the complex exponentiated temperature plane for statistical mechanical models (for a recent list see, e.g., Martin (1985a, b)). Unfortunately with the exception of the Ising model (Onsager 1944) results have been restricted to the finite lattice. In this paper we provide generic examples in which the thermodynamic limit is taken in one dimension and explain the relationship with the eigenvalues of the transfer matrix, their degeneracy and analytic or sheet structure (see, e.g., Phillips 1957) in the complex plane.

These examples were obtained using some remarkable block diagonalisation properties of the transfer matrix for the staggered ice model representation of the q-state Potts model (Baxter *et al* 1976). These properties are independently interesting since they explain the failure of the Bethe ansatz approach (Baxter 1982a) to solve the model for $q=1, 2$.

In this paper 'block diagonalisation' of the transfer matrix will mean block diagonalisation by temperature-independent similarity transformations, which preserve the polynomial nature (see below) of matrix elements. A 'dominant sub-block' is one containing the subset of eigenvalues, one or another of which has the largest absolute magnitude at every point in the complex plane.

The equivalence between the Potts model and the staggered ice model is well described in the literature (see Baxter (1982a), for instance). In the present paper we use the conventional Potts Hamiltonian

$$H=-\beta \sum_{\substack{\text{lattice}\\ \text{links}\\ ij}} \delta(\sigma_i, \sigma_j)$$

(where σ_i are q-valued site variables) and we use the variable $y=\exp(\beta)$, the ice model variable $X=(y-1)/\sqrt{q}$, or the renormalised variable $\bar{x}=X/\sqrt{q}$ interchangeably, depending on conditions of brevity and clarity.

In the next section we present selected examples in a pedagogical format. We then return to explain how the block diagonalisation of the transfer matrix necessary for this presentation is achieved using new results on the staggered ice model. The exact solutions we obtain are written down explicitly in an appendix.

2. Examples

The first example we consider is a four-site-wide strip (medial lattice) staggered ice model or q-state Potts model (Baxter *et al* 1976). The lattice is periodic in the four-site direction and, after a minor modification to the Baxter *et al* formulation (see later), translation symmetry and line conservation reduce the 16×16 two-layer transfer matrix (a straightforward generalisation of the homogeneous ice model version described by Baxter (1982a)) to an essential 4×4 matrix. This block diagonalises further after some work. The relevant eigenvalues for the Potts model lie in a 2×2 block with

$$\lambda_{\pm}=A_4\pm\sqrt{B_4} \tag{1}$$

where A_4 and B_4 (the subscripts refer to lattice size) are dichromatic polynomials in q and the exponentiated temperature variable (see the appendix).

The largest eigenvalue on the real axis for $q=4$ is λ_+, so in this case the free energy is (see, for example, Schultz *et al* 1964)

$$\lim_{N\to\infty}\frac{\ln Z_{4\times N}}{N}=\lim_{N\to\infty}\frac{1}{N}\ln(\lambda_+^N+\lambda_-^N)=\ln(\lambda_+). \tag{2}$$

In order to make the zero distribution of the partition function Z manifest we invoke the easily proven identity

$$\lambda_1^N+\lambda_2^N=\prod_{\substack{n=1,N\\(n=\frac{1}{2},N-\frac{1}{2}\text{ for }N\text{ even})}}\left[\lambda_1+\exp\left(\frac{2\pi i n}{N}\right)\lambda_2\right]$$
$$\sim\prod_{n=1,N/2}\left[\lambda_1^2+\lambda_2^2+2\cos\left(\frac{2\pi n}{N}\right)\lambda_1\lambda_2\right] \tag{3}$$

for any λ_1, λ_2. So

$$\ln\lambda_+=\frac{1}{2\pi}\int_0^{\pi}\ln\{2[(A_4)^2+B_4]+2\cos p\,[(A_4)^2-B_4]\}\,\mathrm{d}p \tag{4}$$

which is the logarithm of an infinite polynomial with lines of zeros (on the loci $|\lambda_+|=|\lambda_-|$) terminating at the zeros of the algebraic determinant B_4 (see also Wood 1985).

Notice that the zeros for finite N lie on the same loci (equation (3)). Thus in a non-block diagonalised finite lattice case it is always possible, in principle, to choose boundary conditions for which the zeros still lie on these loci. In other cases the zeros converge to the loci as $N\to\infty$, since

$$Z_N=C_+\lambda_+^N+C_-\lambda_-^N+\sum_n C_n\lambda_n^N=0 \tag{5}$$

(where $C_+=C_-$ since Z is polynomial) when

$$\left(\frac{\lambda_-}{\lambda_+}\right)^N=-1-\sum_n\frac{C_n}{C_+}\left(\frac{\lambda_n}{\lambda_+}\right)^N.$$

The latter terms tend to zero as $N \to \infty$ and the identity is then satisfied when $\lambda_-/\lambda_+ = \exp[(i\pi(2n+1)/N)]$ (integer n). This shows that finite lattice results can be used to give an image of the limiting distribution, in the sense that a finite set of points distributed on a smooth locus give an image of that locus.

For example, figure 1 shows the zeros of the partition function for a $q=4$-state 4×32 medial lattice model (from Martin 1985b). Deviations from the limiting loci (from equation (4)) are undetectable, and in particular the endpoints of the distributions are indistinguishable from the zeros of B_4 on this scale (except for tiny deviations visible at the left-hand ends of the unit circle loci).

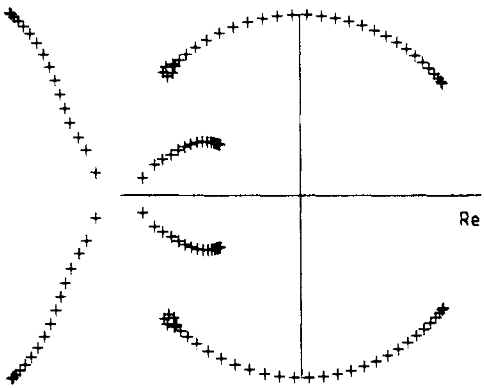

Figure 1. Zeros of the partition function in the complex X plane for a 4×32 medial lattice $q=4$ Potts model. In this and subsequent figures the scale is set by unit length of the positive real axis.

The preceding analysis works for any model whose largest absolute magnitude eigenvalue at any point in the complex plane is always either one or other of a subset of two eigenvalues of the transfer matrix. In these cases it defines the relationship between zeros of the partition function, analyticity of the free energy and degeneracy of eigenvalues of the transfer matrix.

Notice (for later use) that if $\lambda_1 = C$, $\lambda_2 = D$ with C and D polynomials in equation (3), then the logarithm of the infinite polynomial is

$$\frac{1}{2\pi}\int_0^{\pi} \ln(C^2+D^2+2\cos p\, CD)\,\mathrm{d}p = \begin{cases} \ln C & \text{where } |C|>|D| \\ \ln D & \text{where } |D|>|C| \end{cases} \qquad (6)$$

and the zeros of the infinite polynomial form the (closed) boundary between these two regions.

We have seen that finite lattice results give an image of the limiting distribution. Now we can alternatively regard the partition function

$$Z_{M\times N} = \langle\alpha|(T_M)^N|\beta\rangle \qquad (7)$$

where T_M is the M-site periodic transfer matrix and $\langle\alpha|$ and $|\beta\rangle$ represent some boundary conditions, as

$$Z_{M\times N} = \mathrm{Tr}[(T_N^{\alpha\beta})^M] \qquad (8)$$

(where $T_N^{\alpha\beta}$ is the N-site matrix with boundary conditions given by $\langle\alpha|$ and $|\beta\rangle$). From this point of view then, the present $N = \infty$ zero distribution gives an image of the full thermodynamic limit distribution (which improves as M is increased). Of course, the analytic structure of eigenvalues of T_N will in general be more complicated than the present case, as we will see in the next example.

For a six-site-wide strip (and $q = 4$) the largest eigenvalues of the 64×64 two-layer transfer matrix are in a 3×3 block:

$$\lambda_{\substack{e\\a\\a^2}} = A_6 + \substack{1\\a\\a^2}\,(C_6 + \sqrt{B_6})^{1/3} + \substack{1\\a^2\\a}\,(C_6 - \sqrt{B_6})^{1/3} \qquad (9)$$

where $a^3 = 1$ and A_6, B_6 and C_6 are polynomials (see the appendix). The largest eigenvalue on the real axis is λ_e so

$$\lim_{N\to\infty} \frac{\ln Z_{6\times N}}{N} = \lim_{N\to\infty} \frac{1}{N} \ln(\lambda_e^N + \lambda_a^N + \lambda_{a^2}^N) = \ln \lambda_e. \qquad (10)$$

It is easy to see that this is the logarithm of an infinite polynomial (but not so easy to explicitly represent it in this way). It is also easy to show that the zeros of the polynomial lie on loci corresponding to the degenerate absolute magnitude of the largest two eigenvalues (a straightforward generalisation of the two-sheet case above) and terminating at zeros of the algebraic determinant B_6.

For finite N the zeros do not lie exactly on the locus due to the presence of the smaller third eigenvalue. For $N = 32$, however, the discrepancy is already very small (undetectable on the scale of figure 2, for example, where the relevant zeros of B_6 in X at $q = 4$ are again indistinguishable from the endpoints of the distributions of zeros of the partition function for a 6×32 lattice).

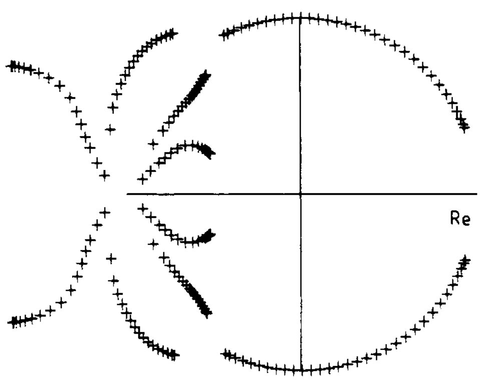

Figure 2. Zeros for a 6×32 $q = 4$ Potts model.

The same analysis goes through for $q > 4$ (see the appendix) and for any model whose largest absolute magnitude eigenvalue at any point in the complex plane is always one or another of a subset of three eigenvalues of the transfer matrix.

Results for $q = 4$ on 8×24 and 10×16 lattices are given in figures 3 and 4. The argument following equation (5) can be generalised to any number of sheets and comparison with other lattice sizes in the long direction suggest these results to be well converged to the loci of degenerate absolute magnitude of the largest eigenvalues. The number of branch points (which are zeros of the algebraic determinant) should not be confused with the number of sheets (equivalent to the number of distinct largest eigenvalues over the complex plane).

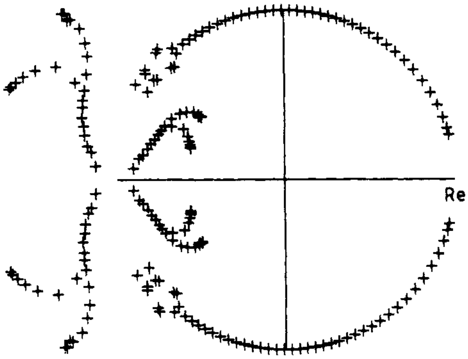

Figure 3. Zeros for an 8×24 $q=4$ Potts model.

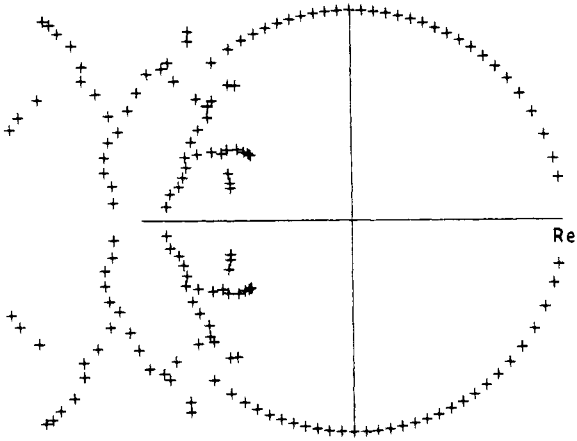

Figure 4. Zeros for a 10×16 $q=4$ Potts model.

We have not yet discussed how our diagonalised examples were obtained. We will see that they appear naturally following the resolution of some boundary problems for the $0<q<4$ cases discussed in the next section.

3. Method

It has long been puzzled over (see, for example, Baxter 1982a) that the staggered ice model has not been solved at $q=2$ (the Ising model) or even at $q=1$. The problem is that the ice model spectrum at $q=1,2$ has some eigenvalues which dominate in a finite region of the complex plane, but which are not associated with the Potts model (Baxter 1982b).

The simplest $q=2$ example is the cyclicly bounded two-site-wide strip. The largest eigenvalue is contained in a 2×2 block T_2, from which one readily obtains

$$ST_2S^{-1} = \begin{pmatrix} (y+\mathrm{i})^2(y-\mathrm{i})^2 & 0 \\ 0 & 4(y-1)^2 \end{pmatrix} \tag{11}$$

where S is a constant matrix.

Now transfer matrices are usually formulated directly from an exponentiated Hamiltonian so that for real temperatures and finite lattices the matrix is positive

(Bellman 1960) and (by Perron's theorem and analyticity) has a unique largest eigenvalue which is the same function over the whole real line. In the ice model representation (for $0<q<4$) the matrix is not positive. In fact the transfer matrix (11) gives a free energy as in equation (6), which describes a model with a phase transition at the antiferromagnetic critical point (and its unphysical dual—the double zeros of the algebraic determinant are at $y=-1\pm\sqrt{2}$ and $1\pm i\sqrt{2}$) but otherwise looks nothing like the Ising model.

Considering the size of the system in the periodic direction some discrepancy from the Ising model is not surprising. However, a finite lattice calculation in which the non-trivial Potts boundary conditions (Baxter *et al* 1976, Baxter 1982a) are applied gives

$$\langle\alpha|(T_2)^N|\beta\rangle_{\text{Potts}}=((y+i)(y-i))^{2N} \tag{12}$$

whereas $\text{Tr}[(T_2)^N]$, of course, gives a finite lattice version of the boundary in equation (6).

In this case, then, boundary conditions are not irrelevant in the thermodynamic limit (cf Baxter *et al* 1976). Specifically we have that

$$S|\beta\rangle_{\text{Potts}}=\begin{pmatrix}1\\0\end{pmatrix}. \tag{13}$$

So, as we have seen, any finite lattice zero distribution (from a given matrix) converges to the same pattern for sufficiently large lattices independent of boundary conditions unless, as here, part of a dominant sub-block is projected out.

Now a discrepancy between large finite lattice distributions using Potts and periodic boundary conditions occurs again in the four-site strip case. Since the Potts bra and ket are trivial in X (Baxter 1982a) this again implies block diagonalisability of the transfer matrix (i.e. preserving polynomial matrix elements). Furthermore, the results obtained using Potts boundary conditions give zeros lying exactly on the Onsager locus (Fisher 1964). This is surprisingly rare for finite lattice results (Abe and Katsura 1970, Maillard and Rammal 1983) and suggests that the block projected out by the Potts boundaries is 2×2 (see above and, e.g., Stanley (1971)).

Armed with this information it is relatively easy to construct by inspection a constant similarity transformation matrix S_4 which block diagonalises the essential 4×4 transfer matrix T_4 ($S_4\ T_4\ S_4^{-1}=\tilde{T}_4(2)$ where $\tilde{T}_4(2)$ is a block diagonal matrix given in the appendix), and in fact gives

$$S_4|\beta_4\rangle_{\text{Potts}}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}. \tag{14}$$

Generalising the elements of S_4 to simple functions of q we quickly find that the full q-dependent transfer matrix block diagonalises in the same way (see the appendix). For $q>4$ Perron's theorem applies to the full transfer matrix and results using Potts and periodic boundaries converge to the same Potts model partition function (i.e. the Potts sub-block is dominant). For $0<q<4$, however, the situation is similar to that for $q=2$.

Remarkably it turns out that the real (double) zeros of the algebraic determinant for the non-Potts sub-block give Baxter's antiferromagnetic critical curves (Baxter 1982b—see later). The points of degeneracy of the largest eigenvalue from each block (the points of intersection of the analytic boundary of the free energy for the full ice model with the real axis) are also very close to Baxter's curves.

Once again for a six-site strip the $q=2$ finite lattice results indicate a 2×2 block projected out by the Potts bounds. Again it is relatively easy to construct a similarity transformation for the transfer matrix T_6 which manifests this property. In this case, however, a generalised transformation only block diagonlises the q-dependent matrix to a 3×3 block. This is given in the appendix.

Finite lattice results for larger lattices (e.g. eight and ten sites in the short direction) continue to demonstrate the block diagonalisability of the transfer matrix. Compare the periodic bound result for $q=3$ on a 10×20 lattice (figure 5) with Potts bound results (figure 6 and, for example, Martin (1985a)). In the $q=2$ case such results

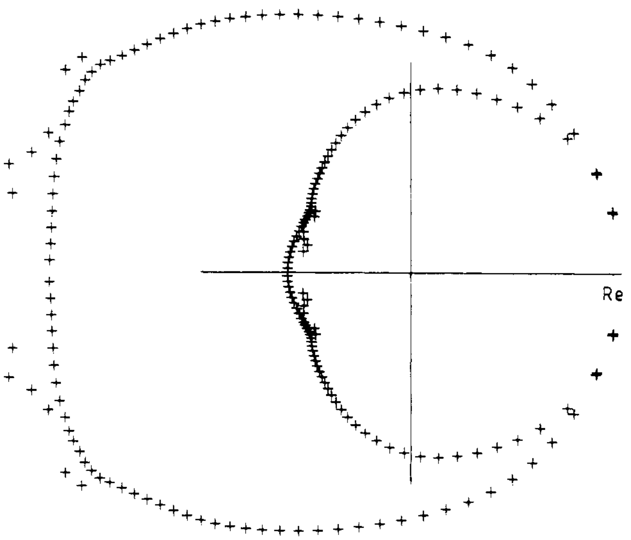

Figure 5. Zeros for a 10×20 ice model representation periodically bounded $q=3$ staggered ice model.

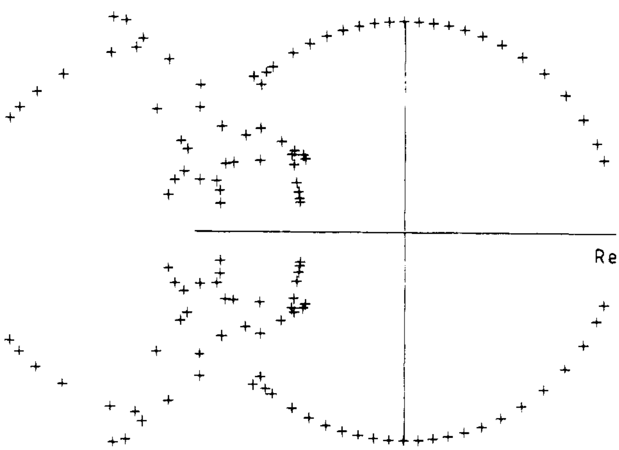

Figure 6. Zeros for an 8×16 ice model representation Potts bounded (Baxter *et al* 1976) $q=3$ Potts model.

continue to suggest that the Potts block is 2×2 (see Martin 1985b). Also, although we have not used the result in this section, it is worth noting that the Potts block in the $q=1$ case is 1×1 (see the appendix).

4. Discussion

When Blöte *et al* (1981) and Blöte and Nightingale (1982) studied the q-state model numerically they used the Whitney polynomial approach (see their papers for references). However, for analytical work it seems that the present construction is preferable. For example Maillard (1985) gives a 3×3 matrix for the three-site conventional lattice strip which is effectively only fractionally larger than the four-site medial lattice strip for which the transfer matrix is 2×2. The key seems to be that the ice model representation has very favourable edge properties in the short direction.

The ice model representation we use is that of Baxter *et al* (1976). However, their weighted seam procedure for the cyclicly bounded case obfuscates the translational symmetry of the transfer matrix. In the present work we associated an extra weight $e^{\theta/M}(e^{-\theta/M})$ with every clockwise (anticlockwise) pointing arrow (i.e. in the periodic direction) on the medial lattice (where M is the width of the lattice in the periodic direction and $\cosh\theta=\sqrt{q}/2$). This does not alter the partition function, but greatly facilitates the block diagonalisation of the transfer matrix.

Noting the limitations of the connection between the full ice model partition function and the Potts model (for $0<q<4$) it is important to remember that Baxter's critical curves (Baxter 1982a, b) were derived by this route. Fortunately a finite lattice analysis of two- to ten-site-wide strip lattices shows that a neighbourhood of the ferromagnetic critical point is entirely in the region dominated by the Potts subspace. This is not true in general of the antiferromagnetic region, which shows why Baxter's curves contain more information than the Potts phase transition points (e.g. at $q=1$).

A recent letter by Wood (1985) conjectures the use of results such as those presented in this paper for obtaining exact information on the zero distribution in the full thermodynamic limit. The conjecture does not appear to be supported, however, inasmuch as the set of zeros for one semi-infinite lattice is not in general a subset of the zeros for a semi-infinite lattice of greater width (e.g. $q=4$ in this paper). The Ising model is a special case because its zero distribution may already be determined on symmetry grounds (see, e.g., Martin 1985a).

In this paper we have established the existence of a sub-block addressed by the Potts boundary conditions. It may now be possible to formalise the diagonalisation procedure and solve the model in this subspace with a Bethe ansatz. For $q=1$ and $q=2$ it is simply a matter of solving an incomplete set of simultaneous equations for the (similarity transformation) matrices which appropriately localise the boundary vectors.

Our exact results (well represented, for instance, by the images of figure 1 and figure 2) disprove the conjecture of Maillard and Rammal (1983) that the zero distribution for the $q\geqslant4$ Potts model is restricted to the circle $|X|=1$.

Acknowledgments

I would like to thank John Martin for inspiration.

Appendix

For the sake of brevity in the present paper we have been largely unconcerned with the task of extracting statistical mechanical information from our specific results. Instead we include the exact solutions for the four-site and six-site medial lattice strip q-state models for reference.

The dominant sub-block of the four-site strip two-layer transfer matrix for general q may be written

$$\tilde{T}_4(q)=\begin{pmatrix} 4q\bar{x}^4+(q-1)m_{11} & (q-1)m_{12} & 0 & 0 \\ m_{21} & (1+\bar{x})^8+(q-1)m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix}$$

where the Potts sub-block is given by

$$m_{11}=32\bar{x}^5+24q\bar{x}^6+8q^2\bar{x}^7+q^3\bar{x}^8$$

$$m_{12}=-(\bar{x}^2+6\bar{x}^3+\tfrac{1}{2}(q+32)\bar{x}^4+2(3q+8)\bar{x}^5+q(q+12)\bar{x}^6+4q^2\bar{x}^7+\tfrac{1}{2}q^3\bar{x}^8)$$

$$m_{21}=4m_{12}+16q\bar{x}^7+2q^3\bar{x}^8$$

$$m_{22}=6\bar{x}^4+24\bar{x}^5+4(q+7)\bar{x}^6+8(q+1)\bar{x}^7+(q^2+q+1)\bar{x}^8$$

(the Potts boundary conditions pick out $(1+\bar{x})^8$ at $q=1$), and the non-Potts block by

$$m_{33}=4\bar{x}^2+20\bar{x}^3+(q+32)\bar{x}^4+4q\bar{x}^5$$

$$m_{34}=16\bar{x}^2+4(q+16)\bar{x}^3+32q\bar{x}^4+4q^2\bar{x}^5$$

$$m_{43}=\bar{x}^3+8\bar{x}^4+(q+16)\bar{x}^5+4q\bar{x}^6$$

$$m_{44}=4\bar{x}^3+(q+32)\bar{x}^4+20q\bar{x}^5+4q^2\bar{x}^6.$$

Some simple algebra then gives the objects (A_4, B_4, etc) discussed in the text.

The Potts sub-block of the six-site strip two-layer transfer matrix for general q may be written

$$\begin{pmatrix} 4(1+\bar{x})^{12}+(q-1)n_{11} & (q-1)n_{12} & (q-1)n_{13} \\ (q-2)n_{21} & n_{22} & n_{23} \\ (q-2)n_{31} & n_{32} & n_{33} \end{pmatrix}$$

where

$$\begin{aligned} n_{11}=-(4&+48\bar{x}+(12q+240)\bar{x}^2+(120q+640)\bar{x}^3+(588q+804)\bar{x}^4 \\ &+(24q^2+1776q-480)\bar{x}^5+q(320q+1040)\bar{x}^6+(24q^2+1776q-480)q\bar{x}^7 \\ &+(588q+804)q^2\bar{x}^8+(120q+640)q^3\bar{x}^9 \\ &+(12q+240)q^4\bar{x}^{10}+48q^5\bar{x}^{11}+4q^6\bar{x}^{12}) \\ &+4(66\bar{x}^2+220\bar{x}^3+495\bar{x}^4+(q+1)(792\bar{x}^5+924\bar{x}^6) \\ &+(q^2+q+1)(792\bar{x}^7+495\bar{x}^8)+(q^3+q^2+q+1)220\bar{x}^9 \\ &+(q^4+q^3+q^2+q+1)(66\bar{x}^{10}+12\bar{x}^{11})+(q^5+q^4+q^3+q^2+q+1)\bar{x}^{12}) \end{aligned}$$

and the non-zero coefficients C_{ij}^{lm} in

$$n_{lm}=\sum_{ij}\bar{x}^i q^{j-2} C_{ij}^{lm}$$

Table 1. The coefficients C_{ij}^{lm} in $n_{lm} = \Sigma_{ij}\bar{x}^i q^{j-2} C_{ij}^{lm}$.

i	j	C_{ij}^{12}	i	j	C_{ij}^{13}	i	j	C_{ij}^{21}	i	j	C_{ij}^{22}	i	j	C_{ij}^{23}	i	j	C_{ij}^{31}	i	j	C_{ij}^{32}	i	j	C_{ij}^{33}
0	2	12	0	2	−32	2	2	−8	2	2	−24	2	2	64	0	2	1	0	2	3	0	2	−8
1	2	144	1	2	−384	3	2	−80	2	3	24	2	3	−64	1	2	12	0	3	−3	0	3	8
2	2	756	2	2	−2016	4	2	−416	3	2	−240	3	2	640	2	2	60	1	2	36	1	2	−96
2	3	24	2	3	−48	4	3	−36	3	3	240	3	3	−640	2	3	3	1	3	−36	1	3	96
3	2	2280	3	2	−6080	5	2	−896	4	2	−1248	4	2	3328	3	2	160	2	2	180	2	2	−480
3	3	240	3	3	−480	5	3	−496	4	3	1092	4	3	−2848	3	3	30	2	3	−174	2	3	468
4	2	4176	4	2	−11136	6	3	−1984	4	4	84	4	4	−192	4	2	192	2	4	−6	2	4	12
4	3	1212	4	3	−2448	6	4	−168	5	2	−2688	5	2	7168	4	3	132	3	2	480	3	2	−1280
5	2	4032	5	2	−10752	7	3	−896	5	3	1296	5	3	−3392	5	3	240	3	3	−420	3	3	1160
5	3	3912	5	3	−7968	7	4	−1456	5	4	1128	5	4	−2528	5	4	6	3	4	−60	3	4	120
5	4	48	5	4	−96	7	5	−24	6	3	−4400	6	3	10368	7	4	−240	4	2	576	4	2	−1536
6	3	7776	6	3	−15552	8	4	−1184	6	4	4184	6	4	−8960	7	5	−6	4	3	−336	4	3	1116
6	4	672	6	4	−1344	8	5	−564	6	5	352	6	5	−736	8	4	−192	4	4	−267	4	4	528
7	3	4032	7	3	−5376	9	5	−720	7	3	−2688	7	3	3584	8	5	−132	5	3	384	5	3	−384
7	4	3912	7	4	−7680	9	6	−120	7	4	−240	7	4	2816	9	5	−160	5	4	−480	5	4	864
7	5	48	7	5	−96	10	6	−248	7	5	3048	7	5	−6208	9	6	−30	5	5	−12	5	5	24
8	4	4176	8	4	−5568	10	7	−12	7	6	48	7	6	−96	10	6	−60	7	4	−384	6	3	784
8	5	1212	8	5	−2400	11	7	−48	8	4	−3552	8	4	4736	10	7	−3	7	5	480	6	4	−296
9	5	2280	9	5	−3040	12	8	−4	8	5	2436	8	5	−2432	11	7	−12	7	6	12	6	5	−16
9	6	240	9	6	−480				8	6	1152	8	6	−2304	12	8	−1	8	4	−576	7	4	1152
10	6	756	10	6	−1008				9	5	−2160	9	5	2880				8	5	336	7	5	−1056
10	7	24	10	7	−48				9	6	1920	9	6	−2400				8	6	267	7	6	−24
11	7	144	11	7	−192				9	7	240	9	7	−480				9	5	−480	8	4	768
12	8	12	12	8	−16				10	6	−744	10	6	992				9	6	420	8	5	−228
									10	7	720	10	7	−944				9	7	60	8	6	−540
									10	8	24	10	8	−48				10	6	−180	9	5	640
									11	7	−144	11	7	192				10	7	174	9	6	−520
									11	8	144	11	8	−192				10	8	6	9	7	−120
									12	8	−12	12	8	16				11	7	−36	10	6	240
									12	9	12	12	9	−16				11	8	36	10	7	−228
																		12	8	−3	10	8	−12
																		12	9	3	11	7	48
																					11	8	−48
																					12	8	4
																					12	9	−4

are given in table 1. Potts boundary conditions pick out the lower block at $q = 2$ and the upper element at $q = 1$ in this representation. Again the objects discussed in the text (A_6, B_6, etc) may be simply derived from the above.

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